Chapter Highlights

1. **The Grand Cube of Theoretical Physics**
   The 'big picture' of theoretical physics can be nicely summarized in terms of a unit cube made of the fundamental constants $G, \hbar, c^{-1}$ representing the three axes. The vertices and linkages of this cube — which we will explore in different chapters of this book — allow you to appreciate different phenomena and their inter-relationships. This chapter introduces the Cube of Theoretical Physics and relates it to the rest of the book.

2. **The Emergence of Classical Physics**
   Quantum physics works with probability amplitudes while classical physics assumes deterministic evolution for the dynamical variables. For example, in non-relativistic quantum mechanics, you will solve the Schrodinger equation in a potential to obtain the wave function $\psi(t,q)$, while the same problem — when solved classically — will lead to a trajectory $q(t)$. How does a deterministic trajectory arise from the foggy world of quantum uncertainty? We will explore several aspects of this correspondence in this chapter, some of which are nontrivial. You will discover the *real* meaning of the Hamilton-Jacobi equation (without the usual canonical transformations, generating functions and other mumbo-jumbo) and understand why the Hamilton-Jacobi equation told us $p_a = \partial_a S = (-\partial_t S, \nabla S) = (E, \boldsymbol{p})$ even before the days of four vectors and special relativity. We will also address the question of why the Lagrangian is equal to kinetic energy minus potential energy (or is it, really?) and why there are *only* two classical fields, electromagnetism and gravity. In fact, you will see that classical physics makes better sense as a limit of quantum physics!

3. **Orbits of Planets *are* Circles!**
   The orbits of planets, or any other body moving under an inverse square law force, can be understood in a simple manner using the idea of the velocity space. Surprisingly, a particle moving in an ellipse, parabola or a hyperbola in real space moves in a circle in the velocity space. This approach allows you to solve the Kepler problem in just two steps! We will also explore the peculiar symmetry of the Lagrangian that leads to the conservation of the Runge-Lenz vector and the geometrical insights that it provides. Proceeding to the relativistic versions of Kepler/Coulomb problem you will discover why the forces *must be* velocity dependent in a relativistic theory and describe a new feature in the special relativistic Coulomb problem, viz. the existence of orbits spiraling to the center.

4. **The Importance of being Inverse-square**
   This chapter continues the exploration started in the previous one. The Coulomb problem, which corresponds to motion in a potential that varies as r^{-1}, has a peculiar symmetry which leads to a phenomenon known as 'accidental' degeneracy. This feature exists both in the classical and quantum domains and allows some interesting, alternative ways to understand, e.g., the hydrogen atom spectrum. We will see how one can find the energy levels of the hydrogen atom without solving the Schrodinger equation and how to map the 3D Coulomb problem to a 4D harmonic oscillator problem. The $(1/r)$ nature of the potential also introduces several peculiarities in the *scattering* problem and we will investigate the questions: (i) How come quantum Coulomb scattering leads *exactly* to the Rutherford formula? What happened to the $\hbar$? (ii) How come the Born approximation gives the exact result for the Coulomb potential? What do the 'unBorn' terms contribute?!

5. **Potential surprises in Newtonian Gravity**
   How unique is the distribution of matter which will produce a given Newtonian gravitational field in a region of space? For example, can a non-spherical distribution of matter produce a strictly inverse square force outside the source? Can a non-planar distribution of matter produce a strictly constant gravitational force in some region? We discuss the rather surprising answers to these questions in this chapter. It turns out that the relation between the density distribution and the gravitational force is far from what one would have naively imagined from the textbook examples.

6. **Lagrange and his Points**
   A solution to the 3-body problem in gravity, due to Lagrange, has several remarkable features. In particular, it describes a situation in

which a particle, located at the *maxima* of a potential, remains stable against small perturbations. We will learn a simple way of obtaining this equilateral solution to the three body problem and understanding its stability.

7. **Getting the most of it!**
   Extremum principles play a central role in theoretical physics in many guises. We will discuss, in this chapter, some curious features associated with a few unusual variational problems. We start with a simple way to solve the standard brachistochrone problem and address the question: How come the cycloid solves all the chron-ic problems? (Or does it, really?). We then consider the brachistochrone problem in a real, $(1/r^2)$, gravitational field and describe a new feature which arises: viz. the existence of a forbidden zone in space not accessible to brachistochrone curves! We will also determine the shape of a planet that exerts the maximum possible gravitational force at a point on its surface — a shape which does not even have a name! Finally, we take up the formation of the rainbows with special emphasis on the question: Where do you look for the tertiary (3rd order) rainbow?

8. **Surprises in Fluid Flows**
   The idealized flow of a fluid around a body is a classic text book problem in fluid mechanics. Interestingly enough, it leads to some curious twists and conceptual conundrums. In particular, it leads a surprising divergence which needs to be regularized even in the text book case of fluid flow past a sphere!

9. **Isochronous Curiosities: Classical and Quantum**
   The oscillatory motion of a particle in a one dimensional potential belongs to a class of exactly solvable problems in classical mechanics. This chapter examines some lesser known aspects of this problem in classical and quantum mechanics. It turns out that both $V(x) = ax^2$ and $V(x) = ax^2 + bx^{-2}$ have (1) periods of oscillation which are independent of amplitude in classical physics and (2) equally spaced energy levels in quantum theory. We will explore several features of this curious correspondence. We will also discuss the question of determining the potential from the period of oscillation (in classical physics) or from the energy levels (in quantum physics) which are closely related and clarify several puzzling features related to this issue.

10. **Logarithms of Nature**
    Scaling arguments and dimensional analysis are powerful tools in physics which help you to solve several interesting problems. And when the scaling arguments fail, as in the examples discussed in this

chapter, we are led to a more fascinating situation. A simple example in electrostatics leads to infinities in the Poisson equation and we get a finite $\boldsymbol{E}$ from an infinite ϕ! I also describe the quantum energy levels in the delta function potentials and show how QFT helps you to understand QM better!

11. **Curved Spacetime for pedestrians**
    The spacetime around a spherical body plays a key role in general relativity and is used in the crucial tests of Einstein's theory of gravity. This spacetime geometry is usually obtained by solving Einstein's equations. I will show how this metric can be obtained by a simple — but strange — trick. Along the way, you will also learn a three-step proof as to why gravity must be geometry, the reason why the Lagrangian for a particle in a Newtonian gravitational field is kinetic energy minus potential energy and how to obtain the orbit equation in GR, just from the principle of equivalence.

12. **Black hole is a Hot Topic**
    A fascinating result in black hole physics is that they are not really black! They glow as though they have a surface temperature which arises due to purely quantum effects. I will provide a simple derivation of this hot result based on the interpretation of a plane wave by different observers.

13. **Thomas and his Precession**
    Thomas precession is a curious effect in special relativity which is purely kinematical in origin. But it illustrates some important features of the Lorentz transformation and possesses a beautiful geometric interpretation. We will explore the physical reason for Thomas precession and its geometrical meaning in this chapter and in the next.

14. **When Thomas met Foucault**
    The Foucault pendulum is an elegant device that demonstrates the rotation of the Earth. We describe a paradox related to the Foucault pendulum and provide a geometrical approach to determine the rotation of the plane of the pendulum. By introducing a natural metric in the velocity space we obtain an interesting geometrical relationship between the dynamics of the Foucault pendulum and the Thomas precession discussed in the previous chapter. This approach helps us to understand both phenomena better.

15. **The One-body Problem**
    You might have thought that the one-body problem in physics is trivial. Far from it! One can look at the free particle in an inertial or a non-inertial frame, relativistically or non-relativistically, in flat or

in curved spacetime, classically or quantum mechanically. All these bring in curious correspondences in which the more exact theory provides valuable insights about the approximate description. I start with the surprising — and not widely appreciated — result that you really can't get a sensible free-particle Lagrangian in non-relativistic mechanics while you can do it in relativistic mechanics. In a similar vein, the solution to the Klein-Gordon equation transforms as a scalar under coordinate transformations, while the solution to the Schrodinger equation does not. These conundrums show that classical mechanics makes more sense as a limiting case of special relativity and the non-relativistic Schrodinger equation is simpler to understand as a limiting case of the relativistic Klein-Gordon equation!

16. **The Straight and Narrow Path of Waves**
Discovering unexpected connections between completely different phenomena is always a delight in physics. In this chapter and the next, we will look at one such connection between two unlikely phenomena: propagation of light and the path integral approach to quantum *field* theory! This chapter introduces the notion of paraxial optics in which we throw away half the solutions and still get useful results! I also describe the role of optical systems and how the humble lens acts as an analog device that performs Fourier transforms. In passing, you will also learn how Faraday's law leads to diffraction of light.

17. **If Quantum Mechanics is the Paraxial Optics, then .....**
The *quantum mechanical* amplitude for a particle to propagate from event to event in spacetime shows some nice similarities with the corresponding propagator for the electromagnetic wave amplitude discussed in the previous chapter. This raises the question: If quantum mechanics is paraxial optics, what is the exact theory you get when you go beyond the paraxial approximation? In the path integral approach to quantum mechanics you purposely avoid summing over *all* the paths while in the path integral approach for a *relativistic* particle you *are forced to* sum over all paths. This fact, along with the paraxial optics analogy, provides an interesting insight into the transition from quantum field theory to quantum mechanics and vice versa! I also describe why combining the principles of relativity and quantum theory *demands* a description in terms of fields.

18. **Make it Complex to Simplify**
Some of the curious effects in quantum theory and statistical mechanics can be interpreted by analytically continuing the time coordinate to purely imaginary values. We explore some of these issues in this chapter. In quantum mechanics, this allows us to determine the properties of ground state from an approximate evaluation of path integrals. In statistical mechanics this leads to an unexpected connection

between periodicity in imaginary time and temperature. The power of this approach can be appreciated by the fact that one can derive the black hole temperature in just a couple of steps using this procedure. Another application of the imaginary time method is to understand phenomena like the Schwinger effect which describes the popping out of particles from the vacuum. Finally, I describe a non-perturbative result in quantum mechanics, called the over-the-barrier-reflection, which is easier to understand using complex paths.

19. **Nothing matters a lot**
The vacuum state of the electromagnetic field is far from trivial. Amongst other things, it can exert forces that are measurable in the lab. This curious phenomenon, known as the *Casimir effect*, is still not completely understood. I describe how the probability distribution for the existence of electromagnetic fields in the vacuum can be understood, just from the knowledge of the quantum mechanics of the harmonic oscillator. This chapter also introduces you to the tricks of the trade in quantum field theory, which are essential to get finite answers from divergent expressions - like to prove that the sum of all positive integers is a negative fraction!

20. **Radiation: Caterpillar becomes Butterfly**
The fact that an accelerated charge will radiate energy is considered an elementary textbook result in electromagnetism. Nevertheless, this process of radiation (and its reaction on the charged particle) raises several conundrums about which technical papers are written even today. In this chapter, we will try to understand how the caterpillar ($1/r^2$, radial field) becomes a butterfly ($1/r$, transverse field) in a simple, yet completely rigorous, manner without the Lienard-Wiechert potentials or other red-herrings. I will also discuss some misconceptions about the validity of $\nabla \cdot \boldsymbol{E} = 4\pi\rho$ for radiative fields with retardation effects.

21. **Photon: Wave and/or Particle**
The interaction of charged particles with blackbody radiation is of considerable practical and theoretical importance. Practically, it occurs in several astrophysical scenarios. Theoretically, it illustrates nicely the fact that one can think of the radiation either as a bunch of photons or as electromagnetic waves and still obtain the same results. We shall highlight some non-trivial aspects of this correspondence in this chapter. In particular we will see how the blackbody radiation leads a double life of being either photons or waves and how the radiative transfer between charged particles and black body radiation can be derived just from a Taylor series expansion (and a little trick)! Finally, I will describe the role of radiation reaction force on charged particles to understand some of these results.

22. **Angular Momentum without Rotation**
Not only mechanical systems, but also electromagnetic fields carry energy and momentum. What is not immediately apparent is that certain static electromagnetic configurations (with no rotation in sight) can also have angular momentum. This leads to surprises when this angular momentum is transfered to the more tangible rotational motion of charged particles coupled to the electromagnetic fields. A simple example described in this chapter illustrates, among other things, how an observable effect arises from the unobservable vector potential and why we can be cavalier about gauge invariance in some circumstances.

23. **Ubiquitous Random Walk**
What is common to the spread of mosquitoes, sound waves and the flow of money? They all can be modeled in terms of random walks! Few processes in nature are as ubiquitous as the random walk which combines extraordinary simplicity of concept with considerable complexity in the final result. In this and the next chapter, we shall examine several features of this remarkable phenomenon. In particular, I will describe the random walk in the velocity space for a system of gravitating particles. The diffusion in velocity space can't go on and on — unlike that in real space — which leads to another interesting effect known as dynamical friction — first derived by Landau in an elegant manner.

24. **More on Random Walks: Circuits and a Tired Drunkard**
We continue our exploration of random walks in this chapter with some more curious results. We discuss the dimension dependence of some of the features of the random walk (e.g., why a drunken man will eventually come home but a drunken bird may not!), describe a curious connection between the random walk and electrical networks (which includes some problems you can't solve by being clever) and finally discuss some remarkable features of the random walk with decreasing step-length, which is still not completely understood and leads to Cantor sets, singularities and the golden ratio — in places where you don't expect to see them.

25. **Gravitational Instability of the Isothermal Sphere**
The statistical mechanics of a system of particles interacting through gravity leads to several counter-intuitive features. We explore one of them, called Antonov instability, in this chapter. I describe why the thermodynamics of gravitating systems is non-trivial and how to obtain the mean-field description of such a system. This leads to a self-gravitating distribution of mass called the isothermal sphere which exhibits curious features both from the mathematical and physical

points of view. I provide a simple way of understanding the stability of this system, which is of astrophysical significance.

26. **Gravity bends electric field lines**
Field lines of a point charge are like radially outgoing light rays from a source. You know that the path of light is bent by gravity; do electric field lines also bend in a gravitational field? Indeed they do, and — in the simplest context of a constant gravitational field — both are bent in the same way. Moreover, both form arcs of circles! The Coulomb potential in a weak gravitational field can be expressed in a form which has unexpected elegance. The analysis leads to a fresh insight about electromagnetic radiation as arising from the weight of electrostatic energy in the rest frame of the charged particle, and also allows you to obtain Dirac's formula for the radiation reaction, in three simple steps.

Notations and Conventions

Most of the notations used in the book are fairly standard. You may want to take note of the following:

1. I use the Gaussian system of units to describe electromagnetic phenomena; however, conversion to SI units is completely straightforward in all the relevant chapters.
2. In chapters involving relativity, the Latin letters $a, b, \ldots$ range over the spacetime indices $0, 1, 2, 3$, while the Greek indices $\alpha, \beta, \ldots$ range over the spatial coordinates $1, 2, 3$ with the notation $\partial_i = (\partial/\partial x^i)$ for coordinate derivatives. When the discussion does not involve relativistic physics, this distiction between Latin and Greek subscripts is not maintained. The signature for the spacetime is $(-,+,+,+)$ with η_{ij} = dia $(-1,1,1,1) = \eta^{ij}$. Units with $c = 1$ are used most of the time though c is re-introduced when required.
3. All through the book (and not only in chapters dealing with relativity) I use the summation convention according to which any index repeated in an algebraic expression is summed over its range of values.
4. In topics dealing with quantum mechanics, I often use units with $\hbar = 1$, re-introducing it into the equations only when relevant.
5. In the equations, you will sometimes find the use of the symbol $\equiv$. This indicates that the equation defines a new variable or notation.