Bull. Astr. Soc. India (2007) **35**, 319–338

Higher order normalizations in the generalized photogravitational restricted three body problem with Poynting-Robertson drag

B. S. Kushvah*, J. P. Sharma and B. Ishwar†
University Department of Mathematics, B.R.A. Bihar University Muzaffarpur 842 001, India

Received 23 April 2007; accepted 10 August 2007

Abstract. Higher order normalizations are performed in the generalized photogravitational restricted three body problem with Poynting-Robertson drag. Here we have taken bigger primary as a source of radiation and smaller primary as an oblate spheroid. Whittaker method is used to transform the second order part of the Hamiltonian into the normal form. We have also performed Birkhoff's normalization of the Hamiltonian. For this we have utilized Henrard's method and expanded the coordinates of the infinitesimal body in double D'Alembert series. We have found the values of first and second order components. They are affected by radiation pressure, oblateness and P-R drag. Finally we obtained the third order part of the Hamiltonian zero.

Keywords : celestial mechanics

1. Introduction

The restricted three body problem describes the motion of an infinitesimal mass moving under the gravitational effect of the two finite masses, called primaries, which move in circular orbits around their centre of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The classical restricted three body problem is generalized to include the force of radiation pressure, the Poynting-Robertson effect and oblateness effect.

*email:bskush@hotmail.com
†email:ishwar_ bhola@hotmail.com

Poynting (1903) considered the effect of the absorption and subsequent re-emission of sunlight by small isolated particles in the solar system. His work was later modified by Robertson (1937) who used precise relativistic treatments of the first order in the ratio of the velocity of the particle to that of light. Chernikov (1970) & Schuerman (1980) discussed the position as well as the stability of the Lagrangian equilibrium points when radiation pressure, P-R drag force are included. Murray (1994) systematically discussed the dynamical effect of general drag in the planar circular restricted three body problem. Liou et al. (1995) examined the effect of radiation pressure, P-R drag and solar wind drag in the restricted three body problem.

Moser's conditions (1962), Arnold's theorem (1961) and Liapunov's theorem (1956) played a significant role in deciding the nonlinear stability of an equilibrium point. Moser gave some modifications in Arnold's theorem. Then Deprit & Deprit (1967) investigated the nonlinear stability of triangular points by applying Moser's modified version of Arnold's theorem(1961). Maciejewski & Gozdziewski (1991) described the normalization algorithms of Hamiltonian near an equilibrium point. Niedzielska (1994) investigated the nonlinear stability of the libration points in the photogravitational restricted three body problem. Mishra & Ishwar (1995) studied second order normalization in the generalized restricted problem of three bodies, smaller primary being an oblate spheroid. Ishwar (1997) studied nonlinear stability in the generalized restricted three body problem.

In this paper higher order normalizations are performed in the generalized photogravitational restricted three body problem with Poynting-Robertson drag. Whittaker method is used to transform the second order part of the Hamiltonian into the normal form. We have performed Birkhoff's normalization of the Hamiltonian. For this we have utilized Henrard's method and expanded the coordinates of the third body in double D'Alembert series. We have found the values of first and second order components. The second order components are obtained as solutions of the two partial differential equations. We have employed the first condition of KAM theorem in solving these equations. The first and second order components are affected by radiation pressure, oblateness and P-R drag. Finally we obtained the third order part H_3 of the Hamiltonian in $I_1^{1/2}, I_2^{1/2}$ zero.

2. Location of triangular equilibrium points

Equations of motion are

$$\ddot{x} - 2n\dot{y} = U_x, \quad \text{where,} \quad U_x = \frac{\partial U_1}{\partial x} - \frac{W_1 N_1}{r_1^2} \tag{1}$$

$$\ddot{y} + 2n\dot{x} = U_y, \qquad U_y = \frac{\partial U_1}{\partial y} - \frac{W_1 N_2}{r_1^2} \tag{2}$$

$$U_1 = \frac{n^2(x^2+y^2)}{2} + \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} \tag{3}$$

$$r_1^2 = (x+\mu)^2 + y^2, \quad r_2^2 = (x+\mu-1)^2 + y^2, \quad n^2 = 1 + \frac{3}{2}A_2,$$
$$N_1 = \frac{(x+\mu)[(x+\mu)\dot{x} + y\dot{y}]}{r_1^2} + \dot{x} - ny, \quad N_2 = \frac{y[(x+\mu)\dot{x} + y\dot{y}]}{r_1^2} + \dot{y} + n(x+\mu).$$

$W_1 = \frac{(1-\mu)(1-q_1)}{c_d}$, $\mu = \frac{m_2}{m_1+m_2} \leq \frac{1}{2}$, m_1, m_2 be the masses of the primaries, $A_2 = \frac{r_e^2 - r_p^2}{5r^2}$ be the oblateness coefficient, r_e and r_p be the equatorial and polar radii respectively, r be the distance between primaries, $q_1 = \left(1 - \frac{F_p}{F_g}\right)$ be the mass reduction factor expressed in terms of the particle's radius a, density ρ and radiation pressure efficiency factor χ (in the C.G.S.system) i.e., $q_1 = 1 - \frac{5.6\times10^{-5}\chi}{a\rho}$. Assumption $q_1 = constant$ is equivalent to neglecting fluctuation in the beam of solar radiation and the effect of solar radiation, the effect of the planet's shadow, obviously $q_1 \leq 1$. Triangular equilibrium points are given by $U_x = 0, U_y = 0, z = 0, y \neq 0$, then we have

$$x_* = x_0\left\{1 - \frac{nW_1\left[(1-\mu)\left(1+\frac{5}{2}A_2\right) + \mu(1-\frac{A_2}{2})\frac{\delta^2}{2}\right]}{3\mu(1-\mu)y_0x_0} - \frac{\delta^2}{2}\frac{A_2}{x_0}\right\} \tag{4}$$

$$y_* = y_0\left\{1 - \frac{nW_1\delta^2\left[2\mu - 1 - \mu(1-\frac{3A_2}{2})\frac{\delta^2}{2} + 7(1-\mu)\frac{A_2}{2}\right]}{3\mu(1-\mu)y_0^3} - \frac{\delta^2\left(1-\frac{\delta^2}{2}\right)A_2}{y_0^2}\right\}^{1/2} \tag{5}$$

where $x_0 = \frac{\delta^2}{2} - \mu$, $y_0 = \pm\delta\left(1 - \frac{\delta^2}{4}\right)^{1/2}$ and $\delta = q_1^{1/3}$, as in Kushvah & Ishwar (2006).

3. Normalization of H_2

We used Whittaker (1965) method for the transformation of H_2 into normal form. The Lagrangian function of the problem can be written as

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2+y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} \tag{6}$$
$$+ W_1\left\{\frac{(x+\mu)\dot{x} + y\dot{y}}{2r_1^2} - n\arctan\frac{y}{(x+\mu)}\right\}$$

and the Hamiltonian is $H = -L + p_x\dot{x} + p_y\dot{y}$, where p_x, p_y are the momenta coordinates given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - ny + \frac{W_1}{2r_1^2}(x+\mu), \quad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + nx + \frac{W_1}{2r_1^2}y.$$

For simplicity we suppose $q_1 = 1 - \epsilon$, with $|\epsilon| << 1$ then coordinates of triangular equilibrium points can be written in the form

$$x = \frac{\gamma}{2} - \frac{\epsilon}{3} - \frac{A_2}{2} + \frac{A_2\epsilon}{3} - \frac{(9+\gamma)}{6\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1 \tag{7}$$

$$y = \frac{\sqrt{3}}{2}\left\{1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} + \frac{(1+\gamma)}{9\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1\right\} \tag{8}$$

where $\gamma = 1 - 2\mu$. We shift the origin to L_4. For that, we change $x \to x_* + x$ and $y \to y_* + y$. Let $a = x_* + \mu, b = y_*$ so that

$$a = \frac{1}{2}\left\{1 - \frac{2\epsilon}{3} - A_2 + \frac{2A_2\epsilon}{3} - \frac{(9+\gamma)}{3\sqrt{3}}nW_1 - \frac{8\gamma\epsilon}{27\sqrt{3}}nW_1\right\} \tag{9}$$

$$b = \frac{\sqrt{3}}{2}\left\{1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} + \frac{(1+\gamma)}{9\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1\right\} \tag{10}$$

Expanding L in power series of x and y, we get

$$L = L_0 + L_1 + L_2 + L_3 + \cdots \tag{11}$$

$$H = H_0 + H_1 + H_2 + H_3 + \cdots = -L + p_x\dot{x} + p_y\dot{y} \tag{12}$$

where $L_0, L_1, L_2, L_3 \ldots$ are

$$\begin{aligned} L_0 = {} & \frac{3}{2} - \frac{2\epsilon}{3} - \frac{\gamma\epsilon}{3} + \frac{3\gamma A_2}{4} - \frac{3A_2\epsilon}{2} - \gamma A_2 \\ & - \frac{\sqrt{3}nW_1}{4} + \frac{2\gamma nW_1}{3\sqrt{3}} - \frac{n\epsilon W_1}{3\sqrt{3}} - \frac{23\epsilon\gamma nW_1}{54\sqrt{3}} - nW_1 \arctan\frac{b}{a} \end{aligned} \tag{13}$$

$$\begin{aligned} L_1 = {} & \dot{x}\left\{-\frac{\sqrt{3}}{2} + \frac{\epsilon}{3\sqrt{3}} - \frac{5A_2}{8\sqrt{3}} + \frac{7\epsilon A_2}{12\sqrt{3}} + \frac{4nW_1}{9} - \frac{\gamma nW_1}{18}\right\} \\ & + \dot{y}\left\{\frac{1}{2} - \frac{\epsilon}{3} - \frac{A_2}{8} + \frac{\epsilon A_2}{12} - \frac{\gamma nW_1}{6\sqrt{3}} + \frac{2\epsilon nW_1}{3\sqrt{3}}\right\} \\ & - x\left\{-\frac{1}{2} + \frac{\gamma}{2} + \frac{9A_2}{8} + \frac{15\gamma A_2}{8} - \frac{35\epsilon A_2}{12} - \frac{29\gamma\epsilon A_2}{12} + \frac{3\sqrt{3}nW_1}{8} - \frac{5\epsilon nW_1}{12\sqrt{3}} - \frac{7\gamma\epsilon nW_1}{4\sqrt{3}}\right\} \\ & - y\left\{\frac{15\sqrt{3}A_2}{2} + \frac{9\sqrt{3}\gamma A_2}{8} - 2\sqrt{3}\epsilon A_2 - 2\sqrt{3}\gamma\epsilon A_2 - \frac{nW_1}{8} + \gamma nW_1 - \frac{43\epsilon nW_1}{36} \right. \\ & \left. - \frac{23\gamma\epsilon nW_1}{36}\right\} \end{aligned} \tag{14}$$

$$L_2 = \frac{(\dot{x}^2 + \dot{y}^2)}{2} + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2 + y^2) - Ex^2 - Fy^2 - Gxy \tag{15}$$

$$L_3 = -\frac{1}{3!}\left\{x^3T_1 + 3x^2yT_2 + 3xy^2T_3 + y^3T_4 + 6T_5\right\} \tag{16}$$

where

$$\begin{aligned} E &= \frac{1}{16}\Big\{2-6\epsilon-3A_2-\frac{31A_2\epsilon}{2}-\frac{(69+\gamma)}{6\sqrt{3}}nW_1+\frac{2(307+75\gamma)\epsilon}{27\sqrt{3}}nW_1 \\ &+ \gamma\big\{2\epsilon+12A_2+\frac{A_2\epsilon}{3}+\frac{(199+17\gamma)}{6\sqrt{3}}nW_1-\frac{2(226+99\gamma)\epsilon}{27\sqrt{3}}nW_1\big\}\Big\} \end{aligned} \tag{17}$$

$$\begin{aligned} F &= \frac{-1}{16}\Big\{10-2\epsilon+21A_2-\frac{717A_2\epsilon}{18}-\frac{(67+19\gamma)}{6\sqrt{3}}nW_1+\frac{2(413-3\gamma)\epsilon}{27\sqrt{3}}nW_1 \\ &+ \gamma\big\{6\epsilon-\frac{293A_2\epsilon}{18}+\frac{(187+27\gamma)}{6\sqrt{3}}nW_1-\frac{4(247+3\gamma)\epsilon}{27\sqrt{3}}nW_1\big\}\Big\} \end{aligned} \tag{18}$$

$$\begin{aligned} G &= \frac{\sqrt{3}}{8}\Big\{2\epsilon+6A_2-\frac{37A_2\epsilon}{2}-\frac{(13+\gamma)}{2\sqrt{3}}nW_1+\frac{2(79-7\gamma)\epsilon}{27\sqrt{3}}nW_1 \\ &- \gamma\big\{6-\frac{\epsilon}{3}+13A_2-\frac{33A_2\epsilon}{2}+\frac{(11-\gamma)}{2\sqrt{3}}nW_1-\frac{(186-\gamma)\epsilon}{9\sqrt{3}}nW_1\big\}\Big\} \end{aligned} \tag{19}$$

$$\begin{aligned} T_1 &= \frac{3}{16}\Bigg[\frac{16}{3}\epsilon+6A_2-\frac{979}{18}A_2\epsilon+\frac{(143+9\gamma)}{6\sqrt{3}}nW_1+\frac{(459+376\gamma)}{27\sqrt{3}}nW_1\epsilon \\ &+\gamma\Bigg\{14+\frac{4\epsilon}{3}+25A_2-\frac{1507}{18}A_2\epsilon-\frac{(215+29\gamma)}{6\sqrt{3}}nW_1 \\ &- \frac{2(1174+169\gamma)}{27\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \end{aligned} \tag{20}$$

$$\begin{aligned} T_2 &= \frac{3\sqrt{3}}{16}\Bigg[14-\frac{16}{3}\epsilon+\frac{A_2}{3}-\frac{367}{18}A_2\epsilon+\frac{115(1+\gamma)}{18\sqrt{3}}nW_1-\frac{(959-136\gamma)}{27\sqrt{3}}nW_1\epsilon \\ &+\gamma\Bigg\{\frac{32\epsilon}{3}+40A_2-\frac{382}{9}A_2\epsilon+\frac{(511+53\gamma)}{6\sqrt{3}}nW_1 \\ &- \frac{(2519-24\gamma)}{27\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \end{aligned} \tag{21}$$

$$\begin{aligned} T_3 &= \frac{-9}{16}\Bigg[\frac{8}{3}\epsilon+\frac{203A_2}{6}-\frac{625}{54}A_2\epsilon-\frac{(105+15\gamma)}{18\sqrt{3}}nW_1-\frac{(403-114\gamma)}{81\sqrt{3}}nW_1\epsilon \\ &+\gamma\Bigg\{2-\frac{4\epsilon}{9}+\frac{55A_2}{2}-\frac{797}{54}A_2\epsilon+\frac{(197+23\gamma)}{18\sqrt{3}}nW_1 \\ &- \frac{(211-32\gamma)}{81\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \end{aligned} \tag{22}$$

$$
\begin{aligned}
T_4 &= \frac{-9\sqrt{3}}{16}\Bigg[2-\frac{8}{3}\epsilon+\frac{23A_2}{3}-44A_2\epsilon-\frac{(37+\gamma)}{18\sqrt{3}}nW_1-\frac{(219+253\gamma)}{81\sqrt{3}}nW_1\epsilon \\
&\quad +\gamma\Bigg\{4\epsilon+\frac{88}{27}A_2\epsilon+\frac{(241+45\gamma)}{18\sqrt{3}}nW_1 \\
&\quad -\frac{(1558-126\gamma)}{81\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg]
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
T_5 &= \frac{W_1}{2(a^2+b^2)^3}\Big[(a\dot{x}+b\dot{y})\left\{3(ax+by)-(bx-ay)^2\right\} \\
&\quad - 2(x\dot{x}+y\dot{y})(ax+by)(a^2+b^2)\Big].
\end{aligned} \tag{24}
$$

The second order part H_2 of the corresponding Hamiltonian takes the form

$$
H_2=\frac{p_x^2+p_y^2}{2}+n(yp_x-xp_y)+Ex^2+Fy^2+Gxy \tag{25}
$$

To investigate the stability of the motion, as in Whittaker(1965), we consider the following set of linear equations in the variables x, y:

$$
\begin{aligned}
-\lambda p_x &= \tfrac{\partial H_2}{\partial x} \qquad \lambda x = \tfrac{\partial H_2}{\partial p_x} \\
-\lambda p_y &= \tfrac{\partial H_2}{\partial y} \qquad \lambda y = \tfrac{\partial H_2}{\partial p_y} \\
\text{i.e.}\quad AX &= 0
\end{aligned} \tag{26}
$$

$$
X=\begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix} \quad \text{and} \quad A=\begin{bmatrix} 2E & G & \lambda & -n \\ G & 2F & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{bmatrix}. \tag{27}
$$

Clearly $|A|=0$, implies that the characteristic equation corresponding to Hamiltonian H_2 is given by

$$
\lambda^4+2(E+F+n^2)\lambda^2+4EF-G^2+n^4-2n^2(E+F)=0. \tag{28}
$$

This is characteristic equation whose discriminant is

$$
D=4(E+F+n^2)^2-4\big\{4EF-G^2+n^4-2n^2(E+F)\big\}. \tag{29}
$$

Stability is assured only when $D>0$. i.e

$$
\begin{aligned}
\mu &< \mu_{c_0}-0.221895916277307669\epsilon+2.1038871010983331A_2 \\
&+ 0.493433373141671349\epsilon A_2+0.704139054372097028nW_1 \\
&+ 0.401154273957540929n\epsilon W_1
\end{aligned}
$$

where $\mu_{c_0} = 0.0385208965045513718$. When $D > 0$ the roots $\pm i\omega_1$ and $\pm i\omega_2$ (ω_1, ω_2 being the long/short -periodic frequencies) are related to each other as

$$\begin{aligned}
\omega_1^2 + \omega_2^2 &= 1 - \frac{\gamma\epsilon}{2} + \frac{3\gamma A_2}{2} + \frac{83\epsilon A_2}{12} + \frac{299\gamma\epsilon A_2}{144} - \frac{nW_1}{24\sqrt{3}} + \frac{5\gamma n W_1}{8\sqrt{3}} - \frac{53\epsilon n W_1}{54\sqrt{3}} \\
&\quad - \frac{5\gamma^2 n W_1}{24\sqrt{3}} + \frac{173\gamma\epsilon n W_1}{54\sqrt{3}} - \frac{3\gamma^2\epsilon n W_1}{36\sqrt{3}} \qquad (30) \\
\omega_1^2\omega_2^2 &= \frac{27}{16} - \frac{27\gamma^2}{16} + \frac{9\epsilon}{8} + \frac{9\gamma\epsilon}{8} - \frac{3\gamma^2\epsilon}{8} + \frac{117\gamma A_2}{16} - \frac{241\epsilon A_2}{32} + \frac{2515\gamma\epsilon A_2}{192} \\
&\quad + \frac{35nW_1}{16\sqrt{3}} - \frac{55\sqrt{3}\gamma n W_1}{16} - \frac{5\sqrt{3}\gamma^2 n W_1}{4} - \frac{1277\epsilon n W_1}{288\sqrt{3}} \\
&+ \frac{5021\gamma\epsilon n W_1}{288\sqrt{3}} + \frac{991\gamma^2\epsilon n W_1}{48\sqrt{3}} \qquad (31) \\
&\quad (0 < \omega_2 < \frac{1}{\sqrt{2}} < \omega_1 < 1).
\end{aligned}$$

From (30) and (31) it may be noted that $\omega_j (j = 1, 2)$ satisfy

$$\begin{aligned}
\gamma^2 &= 1 + \frac{4\epsilon}{9} - \frac{107\epsilon A_2}{27} + \frac{2\gamma\epsilon}{3} + \frac{1579\gamma\epsilon A_2}{324} - \frac{25nW_1}{27\sqrt{3}} - \frac{55\gamma n W_1}{9\sqrt{3}} + \frac{3809\epsilon n W_1}{486\sqrt{3}} \\
&+ \frac{4961\gamma\epsilon n W_1}{486\sqrt{3}} + \Big(-\frac{16}{27} + \frac{32\epsilon}{243} + \frac{8\gamma\epsilon}{27} + \frac{208 A_2}{81} - \frac{8\gamma A_2}{27} - \frac{4868\epsilon A_2}{729} - \frac{563\gamma\epsilon A_2}{243} \\
&\quad + \frac{296nW_1}{243\sqrt{3}} - \frac{10\gamma n W_1}{27\sqrt{3}} - \frac{15892\epsilon n W_1}{2187\sqrt{3}} - \frac{1864\gamma\epsilon n W_1}{729\sqrt{3}} \Big)\omega_j^2 \\
&\quad + \Big(\frac{16}{27} - \frac{32\epsilon}{243} - \frac{208A_2}{81} - \frac{1880\epsilon A_2}{729} - \frac{2720nW_1}{2187\sqrt{3}} \\
&+ \frac{49552\epsilon n W_1}{6561\sqrt{3}} - \frac{80\gamma\epsilon n W_1}{2187\sqrt{3}} \Big)\omega_j^4. \qquad (32)
\end{aligned}$$

Alternatively, it can also be seen that if $u = \omega_1\omega_2$, then equation (31) gives

$$\begin{aligned}
\gamma^2 &= 1 + \frac{4\epsilon}{9} - \frac{107\epsilon A_2}{27} - \frac{25nW_1}{27\sqrt{3}} + \frac{3809\epsilon n W_1}{486\sqrt{3}} \\
&+ \gamma\Big(\frac{2\epsilon}{3} + \frac{1579\epsilon A_2}{324} - \frac{55\gamma n W_1}{9\sqrt{3}} + \frac{4961\gamma\epsilon n W_1}{486\sqrt{3}} \Big) \\
&+ \Big(-\frac{16}{27} + \frac{32\epsilon}{243} + \frac{208A_2}{81} - \frac{1880\epsilon A_2}{729} + \frac{320nW_1}{243\sqrt{3}} - \frac{15856\epsilon n W_1}{2187\sqrt{3}} \Big) u^2. \qquad (33)
\end{aligned}$$

Following the method for reducing H_2 into the normal form, as in Whittaker(1965), we use the transformation

$$X = JT \quad \text{where} \quad X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix}, J = [J_{ij}]_{1 \le i,j \le 4}, \ T = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix} \qquad (34)$$

$$P_i = (2I_i\omega_i)^{1/2}\cos\phi_i, \quad Q_i = (\frac{2I_i}{\omega_i})^{1/2}\sin\phi_i, \quad (i = 1, 2). \tag{35}$$

The transformation changes the second order part of the Hamiltonian into the normal form

$$H_2 = \omega_1 I_1 - \omega_2 I_2. \tag{36}$$

The general solution of the corresponding equations of motion are

$$I_i = \text{const.}, \quad \phi_i = \pm\omega_i + \text{const}, \ (i = 1, 2), \tag{37}$$

If the oscillations about L_4 are exactly linear, Eq. (37) represents the integrals of motion and the corresponding orbits are given by

$$\begin{aligned} x &= J_{13}\sqrt{2\omega_1 I_1}\cos\phi_1 + J_{14}\sqrt{2\omega_2 I_2}\cos\phi_2 \qquad (38)\\ y &= J_{21}\sqrt{\frac{2I_1}{\omega_1}}\sin\phi_1 + J_{22}\sqrt{\frac{2I_2}{\omega_2}}\sin\phi_2 + J_{23}\sqrt{2I_1}\omega_1\cos\phi_1 \\ &+ J_{24}\sqrt{2I_2}\omega_2\cos\phi_2 \qquad (39)\end{aligned}$$

where

$$\begin{aligned} J_{13} &= \frac{l_1}{2\omega_1 k_1}\left\{1 - \frac{1}{2l_1^2}\left[\epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}}nW_1 - \frac{(431-3\gamma)}{27\sqrt{3}}nW_1\epsilon\right]\right. \\ &+\frac{\gamma}{2l_1^2}\left[3\epsilon - \frac{29A_2}{36} - \frac{(187+27\gamma)}{12\sqrt{3}}nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] \\ &-\frac{1}{2k_1^2}\left[\frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}}nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\ &-\frac{\gamma}{4k_1^2}\left[\epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}}nW_1 - \frac{(266-93\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\ &\left.+\frac{\epsilon}{4l_1^2k_1^2}\left[\frac{3A_2}{4} + \frac{(33+14\gamma)}{12\sqrt{3}}nW_1\right] + \frac{\gamma\epsilon}{8l_1^2k_1^2}\left[\frac{347A_2}{36} - \frac{(43-8\gamma)}{4\sqrt{3}}nW_1\right]\right\} \qquad (40)\end{aligned}$$

$$\begin{aligned} J_{14} &= \frac{l_2}{2\omega_2 k_2}\left\{1 - \frac{1}{2l_2^2}\left[\epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}}nW_1 - \frac{(431-3\gamma)}{27\sqrt{3}}nW_1\epsilon\right]\right. \\ &-\frac{\gamma}{2l_2^2}\left[3\epsilon - \frac{293A_2}{36} + \frac{(187+27\gamma)}{12\sqrt{3}}nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon\right] \\ &-\frac{1}{2k_2^2}\left[\frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}}nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\ &+\frac{\gamma}{2k_2^2}\left[\epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}}nW_1 - \frac{(268-9\gamma)}{54\sqrt{3}}nW_1\epsilon\right] \\ &\left.-\frac{\epsilon}{4l_2^2k_2^2}\left[\frac{33A_2}{4} + \frac{(1643-93\gamma)}{216\sqrt{3}}nW_1\right] + \frac{\gamma\epsilon}{4l_2^2k_2^2}\left[\frac{737A_2}{72} - \frac{(13+2\gamma)}{\sqrt{3}}nW_1\right]\right\} \qquad (41)\end{aligned}$$

$$\begin{aligned}
J_{21} \;=\; & -\frac{4n\omega_1}{l_1k_1}\left\{1+\frac{1}{2l_1^2}\left[\epsilon+\frac{45A_2}{2}-\frac{717A_2\epsilon}{36}+\frac{(67+19\gamma)}{12\sqrt{3}}nW_1-\frac{(413-3\gamma)}{27\sqrt{3}}nW_1\epsilon\right]\right.\\
&-\frac{\gamma}{2l_1^2}\left[3\epsilon-\frac{293A_2}{36}+\frac{(187+27\gamma)}{12\sqrt{3}}nW_1-\frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon\right]\\
&-\frac{1}{2k_1^2}\left[\frac{\epsilon}{2}-3A_2-\frac{73A_2\epsilon}{24}+\frac{(1-9\gamma)}{24\sqrt{3}}nW_1+\frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon\right]\\
&-\frac{\gamma}{4k_1^2}\left[\epsilon-3A_2-\frac{299A_2\epsilon}{72}-\frac{(6-5\gamma)}{12\sqrt{3}}nW_1-\frac{(268-93\gamma)}{54\sqrt{3}}nW_1\epsilon\right]\\
&\left.+\frac{\epsilon}{8l_1^2k_1^2}\left[\frac{33A_2}{4}+\frac{(68-10\gamma)}{24\sqrt{3}}nW_1\right]+\frac{\gamma\epsilon}{8l_1^2k_1^2}\left[\frac{242A_2}{9}+\frac{(43-8\gamma)}{4\sqrt{3}}nW_1\right]\right\}
\end{aligned} \tag{42}$$

$$\begin{aligned}
J_{22} \;=\; & \frac{4n\omega_2}{l_2k_2}\left\{1+\frac{1}{2l_2^2}\left[\epsilon+\frac{45A_2}{2}-\frac{717A_2\epsilon}{36}+\frac{(67+19\gamma)}{12\sqrt{3}}nW_1-\frac{(413-3\gamma)}{27\sqrt{3}}nW_1\epsilon\right]\right.\\
&-\frac{\gamma}{2l_2^2}\left[3\epsilon-\frac{293A_2}{36}+\frac{(187+27\gamma)}{12\sqrt{3}}nW_1-\frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon\right]\\
&+\frac{1}{2k_2^2}\left[\frac{\epsilon}{2}-3A_2-\frac{73A_2\epsilon}{24}+\frac{(1-9\gamma)}{24\sqrt{3}}nW_1+\frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon\right]\\
&-\frac{\gamma}{4k_2^2}\left[\epsilon-3A_2-\frac{299A_2\epsilon}{72}-\frac{(6-5\gamma)}{12\sqrt{3}}nW_1-\frac{(268-93\gamma)}{54\sqrt{3}}nW_1\epsilon\right]\\
&\left.+\frac{\epsilon}{4l_2^2k_2^2}\left[\frac{33A_2}{4}+\frac{(34+5\gamma)}{12\sqrt{3}}nW_1\right]+\frac{\gamma\epsilon}{8l_2^2k_2^2}\left[\frac{75A_2}{2}+\frac{(43-8\gamma)}{4\sqrt{3}}nW_1\right]\right\}
\end{aligned} \tag{43}$$

$$\begin{aligned}
J_{23} \;=\; & \frac{\sqrt{3}}{4\omega_1l_1k_1}\left\{2\epsilon+6A_2+\frac{37A_2\epsilon}{2}-\frac{(13+\gamma)}{2\sqrt{3}}nW_1+\frac{2(79-7\gamma)}{9\sqrt{3}}nW_1\epsilon\right.\\
&-\gamma\left[6+\frac{2\epsilon}{3}+13A_2-\frac{33A_2\epsilon}{2}+\frac{(11-\gamma)}{2\sqrt{3}}nW_1-\frac{(186-\gamma)}{9\sqrt{3}}nW_1\epsilon\right]\\
&+\frac{1}{2l_1^2}\left[51A_2+\frac{(14+8\gamma)}{3\sqrt{3}}nW_1\right]-\frac{\epsilon}{k_1^2}\left[3A_2+\frac{(19+6\gamma)}{6\sqrt{3}}nW_1\right]\\
&-\frac{\gamma}{2l_1^2}\left[6\epsilon+135A_2-\frac{808A_2\epsilon}{9}-\frac{(67+19\gamma)}{2\sqrt{3}}nW_1-\frac{(755+19\gamma)}{9\sqrt{3}}nW_1\epsilon\right]\\
&-\frac{\gamma}{2k_1^2}\left[3\epsilon-18A_2-\frac{55A_2\epsilon}{4}-\frac{(1-9\gamma)}{4\sqrt{3}}nW_1+\frac{(923-60\gamma)}{12\sqrt{3}}nW_1\epsilon\right]\\
&\left.+\frac{\gamma\epsilon}{8l_1^2k_1^2}\left[\frac{9A_2}{2}+\frac{(34-5\gamma)}{2\sqrt{3}}nW_1\right]\right\}
\end{aligned} \tag{44}$$

$$\begin{aligned}
J_{24} &= \frac{\sqrt{3}}{4\omega_2 l_2 k_2}\Big\{2\epsilon + 6A_2 + \frac{37A_2\epsilon}{2} - \frac{(13+\gamma)}{2\sqrt{3}}nW_1 + \frac{2(79-7\gamma)}{9\sqrt{3}}nW_1\epsilon \\
&\quad -\gamma\left[6 + \frac{2\epsilon}{3} + 13A_2 - \frac{33A_2\epsilon}{2} + \frac{(11-\gamma)}{2\sqrt{3}}nW_1 - \frac{(186-\gamma)}{9\sqrt{3}}nW_1\epsilon\right] \\
&\quad -\frac{1}{2l_2^2}\left[51A_2 + \frac{(14+8\gamma)}{3\sqrt{3}}nW_1\right] - \frac{\epsilon}{k_2^2}\left[3A_2 + \frac{(19+6\gamma)}{6\sqrt{3}}nW_1\right] \\
&\quad -\frac{\gamma}{2l_2^2}\left[6\epsilon + 135A_2 - \frac{808A_2\epsilon}{9} - \frac{(67+19\gamma)}{2\sqrt{3}}nW_1 - \frac{(755+19\gamma)}{9\sqrt{3}}nW_1\epsilon\right] \\
&\quad -\frac{\gamma}{2k_1^2}\left[3\epsilon - 18A_2 - \frac{55A_2\epsilon}{4} - \frac{(1-9\gamma)}{4\sqrt{3}}nW_1 + \frac{(923-60\gamma)}{12\sqrt{3}}nW_1\epsilon\right] \\
&\quad -\frac{\gamma\epsilon}{4l_1^2k_1^2}\left[\frac{99A_2}{2} + \frac{(34-5\gamma)}{2\sqrt{3}}nW_1\right]\Big\} \qquad (45)
\end{aligned}$$

with $l_j^2 = 4\omega_j^2 + 9, (j = 1, 2)$ and $k_1^2 = 2\omega_1^2 - 1, k_2^2 = -2\omega_2^2 + 1$.

4. Second order normalization

In order to perform Birkhoff's normalization, we use Henrard's method (Deprit & Deprit-Bartholomé 1967) for which the coordinates (x, y) of infinitesimal body, to be expanded in double D'Alembert series $x = \sum_{n\geq 1} B_n^{1,0}$, $y = \sum_{n\geq 1} B_n^{0,1}$ where the homogeneous components $B_n^{1,0}$ and $B_n^{0,1}$ of degree n are of the form

$$\sum_{0\leq m\leq n} I_1^{\frac{n-m}{2}} I_2^{\frac{m}{2}} \sum_{(p,q)} C_{n-m,m,p,q}\cos(p\phi_1 + q\phi_2) + S_{n-m,m,p,q}\sin(p\phi_1 + q\phi_2). \qquad (46)$$

The conditions in double summation are (i) p runs over those integers in the interval $0 \leq p \leq n - m$ that have the same parity as $n - m$ (ii) q runs over those integers in the interval $-m \leq q \leq m$ that have the same parity as m. Here I_1, I_2 are the action momenta coordinates which are to be taken as constants of integer, ϕ_1, ϕ_2 are angle coordinates to be determined as linear functions of time in such a way that $\dot\phi_1 = \omega_1 + \sum_{n\geq 1} f_{2n}(I_1, I_2), \dot\phi_2 = -\omega_2 + \sum_{n\geq 1} g_{2n}(I_1, I_2)$ where ω_1, ω_2 are the basic frequencies, f_{2n} and g_{2n} are of the form

$$f_{2n} = \sum_{0\leq m\leq n} {f'}_{2(n-m),2m} I_1^{n-m} I_2^m \qquad (47)$$

$$g_{2n} = \sum_{0\leq m\leq n} {g'}_{2(n-m),2m} I_1^{n-m} I_2^m. \qquad (48)$$

The first order components $B_1^{1,0}$ and $B_1^{0,1}$ are the values of x and y given by (38) and (39). In order to find out the second order components $B_2^{1,0}, B_2^{0,1}$ we consider Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right) - \frac{\partial L}{\partial y} = 0 \qquad (49)$$

$$\text{i.e.}\quad \left.\begin{array}{rcl} \ddot{x} - 2n\dot{y} + (2E - n^2)x + Gy &=& \frac{\partial L_3}{\partial x} + \frac{\partial L_4}{\partial x} \\ \ddot{y} + 2n\dot{x} + (2F - n^2)y + Gx &=& \frac{\partial L_3}{\partial y} + \frac{\partial L_4}{\partial y} \end{array}\right\}. \tag{50}$$

Since x and y are double D'Alembert series, $x^j x^k (j \geq 0, k \geq 0, j + k \geq 0)$ and the time derivatives $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$ are also double D'Alembert series. We can write

$$\dot{x} = \sum_{n\geq 1} \dot{x}_n, \quad \dot{y} = \sum_{n\geq 1} \dot{y}_n, \quad \ddot{x} = \sum_{n\geq 1} \ddot{x}_n, \quad \ddot{y} = \sum_{n\geq 1} \ddot{y}_n$$

where $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$ are homogeneous components of degree n in $I_1^{1/2}, I_2^{1/2}$ i.e.

$$\begin{aligned} \dot{x} &= \frac{d}{dt} \sum_{n\geq 1} B_n^{1,0} = \sum_{n\geq 1} \left[\frac{\partial B_n^{1,0}}{\partial \phi_1} (\omega_1 + f_2 + f_4 + \cdots) \right. \\ &+ \left. \frac{\partial B_n^{1,0}}{\partial \phi_2} (-\omega_2 + g_2 + g_4 + \cdots) \right]. \end{aligned} \tag{51}$$

We write three components $\dot{x}_1, \dot{x}_2, \dot{x}_3$ of $\dot{x}$

$$\dot{x}_1 = \omega_1 \frac{\partial B_1^{1,0}}{\partial \phi_1} - \omega_2 \frac{\partial B_1^{1,0}}{\partial \phi_2} = D B_1^{1,0} \tag{52}$$

$$\dot{x}_2 = \omega_1 \frac{\partial B_2^{1,0}}{\partial \phi_1} - \omega_2 \frac{\partial B_2^{1,0}}{\partial \phi_2} = D B_2^{1,0} \tag{53}$$

$$\begin{aligned} \dot{x}_3 &= \omega_1 \frac{\partial B_3^{1,0}}{\partial \phi_1} - \omega_2 \frac{\partial B_3^{1,0}}{\partial \phi_2} + f_2 \frac{\partial B_1^{1,0}}{\partial \phi_1} - g_2 \frac{\partial B_1^{1,0}}{\partial \phi_2} \\ &= D B_3^{1,0} + f_2 \frac{\partial B_1^{1,0}}{\partial \phi_1} - g_2 \frac{\partial B_1^{1,0}}{\partial \phi_2} \end{aligned} \tag{54}$$

where

$$D \equiv \omega_1 \frac{\partial}{\partial \phi_1} - \omega_2 \frac{\partial}{\partial \phi_2}. \tag{55}$$

Similarly three components $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$ of $\ddot{x}$ are

$$\ddot{x}_1 = D^2 B_1^{1,0}, \quad \ddot{x}_2 = D^2 B_2^{1,0}, \quad \ddot{x}_3 = D^2 B_3^{1,0} + 2\omega_1 f_2 \frac{\partial^2 B_1^{1,0}}{\partial \phi_1^2} - 2\omega_2 g_2 \frac{\partial^2 B_1^{1,0}}{\partial \phi_2^2}.$$

In similar manner we can write the components of $\dot{y}, \ddot{y}$. Putting the values of $x, y, \dot{x}, \dot{y}, \ddot{x}$ and $\ddot{y}$ in terms of double D'Alembert series in equation (50) we get

$$\left(D^2 + 2E - 1 - \frac{3}{2} A_2 \right) B_2^{1,0} - \left\{ 2 \left(1 + \frac{3}{4} A_2 \right) D - G \right\} B_2^{0,1} = X_2 \tag{56}$$

$$\left\{ 2 \left(1 + \frac{3}{4} A_2 \right) D + G \right\} B_2^{1,0} + \left(D^2 + 2F - 1 - \frac{3}{2} A_2 \right) B_2^{0,1} = Y_2 \tag{57}$$

where

$$X_2 = \left[\frac{\partial L_3}{\partial x}\right]_{x=B_1^{1,0}, y=B_1^{0,1}} \quad \text{and} \quad Y_2 = \left[\frac{\partial L_3}{\partial y}\right]_{x=B_1^{1,0}, y=B_1^{0,1}}.$$

These are two simultaneous partial differential equations in $B_2^{1,0}$ and $B_2^{0,1}$. We solve these equations to find the values of $B_2^{1,0}$ and $B_2^{0,1}$, from Eqs (56) and (57)

$$\triangle_1\triangle_2 B_2^{1,0} = \Phi_2, \quad \triangle_1\triangle_2 B_2^{0,1} = -\Psi_2 \quad \text{where} \quad \triangle_1 = D^2+\omega_1^2, \triangle_2 = D^2+\omega_2^2 \tag{58}$$

$$\Phi_2 = (D^2+2F-n^2)X_2 + (2nD-G)Y_2 \tag{59}$$

$$\Psi_2 = (2nD+G)X_2 - (D^2+2E-n^2)Y_2 \tag{60}$$

The Eq.(58) can be solved for $B_2^{1,0}$ and $B_2^{0,1}$ by putting the formula

$$\frac{1}{\triangle_1\triangle_2}\left\{\begin{array}{c}\cos(p\phi_1+q\phi_2)\\ \text{or}\\ \sin(p\phi_1+q\phi_2)\end{array}\right. = \frac{1}{\triangle_{p,q}}\left\{\begin{array}{c}\cos(p\phi_1+q\phi_2)\\ \text{or}\\ \sin(p\phi_1+q\phi_2)\end{array}\right.$$

where

$$\triangle_{p,q} = \left[\omega_1^2-(\omega_1 p-\omega_2 q)^2\right]\left[\omega_2^2-(\omega_1 p-\omega_2 q)^2\right]$$

provided $\triangle_{p,q} \neq 0$. Since $\triangle_{1,0}=0, \triangle_{0,1}=0$ the terms $\cos\phi_1, \sin\phi_1, \cos\phi_2, \sin\phi_2$ are the critical terms, Φ_2 and Ψ_2 are free from such terms. By condition(1) of Moser's theorem $k_1\omega_1+k_2\omega_2 \neq 0$ for all pairs (k_1,k_2) of integers such that $|k_1|+|k_2| \leq 4$, therefore each of $\omega_1, \omega_2, \omega_1 \pm 2\omega_2, \omega_2 \pm 2\omega_1$ is different from zero and consequently none of the divisors $\triangle_{0,0}, \triangle_{0,2}, \triangle_{2,0}, \triangle_{1,1}, \triangle_{1,-1}$ is zero. The second order components $B_2^{1,0}, B_2^{0,1}$ are as follows:

$$\begin{aligned} B_2^{1,0} \;=\;& r_1 I_1 + r_2 I_2 + r_3 I_1\cos 2\phi_1 + r_4 I_2 \cos 2\phi_2 + r_5 I_1^{1/2} I_2^{1/2}\cos(\phi_1-\phi_2)\\ &+r_6 I_1^{1/2} I_2^{1/2}\cos(\phi_1+\phi_2) + r_7 I_1 \sin 2\phi_1 + r_8 I_2 \sin 2\phi_2\\ &+r_9 I_1^{1/2} I_2^{1/2}\sin(\phi_1-\phi_2) + r_{10} I_1^{1/2} I_2^{1/2}\sin(\phi_1+\phi_2) \end{aligned} \tag{61}$$

and

$$\begin{aligned} B_2^{0,1} \;=\;& -\Big\{ s_1 I_1 + s_2 I_2 + s_3 I_1\cos 2\phi_1 + s_4 I_2 \cos 2\phi_2 + s_5 I_1^{1/2} I_2^{1/2}\cos(\phi_1-\phi_2)\\ &+s_6 I_1^{1/2} I_2^{1/2}\cos(\phi_1+\phi_2) + s_7 I_1 \sin 2\phi_1 + s_8 I_2 \sin 2\phi_2\\ &+ s_9 I_1^{1/2} I_2^{1/2}\sin(\phi_1-\phi_2) + s_{10} I_1^{1/2} I_2^{1/2}\sin(\phi_1+\phi_2)\Big\} \end{aligned} \tag{62}$$

where

$$r_1 = \frac{1}{\omega_1^2\omega_2^2}\left\{J_{13}^2\omega_1 F_4 + J_{13}J_{23}\omega_1 F_4' + \left(\frac{J_{21}^2}{\omega_1} + J_{23}^2\omega_1\right)F_4''\right\} \tag{63}$$

$$r_2 = \frac{1}{\omega_1^2\omega_2^2}\left\{J_{14}^2\omega_2 F_4 + J_{14}J_{24}\omega_2 F_4' + \left(\frac{J_{22}^2}{\omega_2} + J_{24}^2\omega_2\right)F_4''\right\} \tag{64}$$

$$r_3 = \frac{-1}{3\omega_1^2(4\omega_1^2-\omega_2^2)}\Bigg\{8\omega_1^3 J_{21}(J_{13}F_1'+2J_{23}F_1'') + 4\omega_1^2\Big[(J_{13}F_2+J_{23}F_2'')J_{13}\omega_1$$
$$- \left(\frac{J_{21}^2}{\omega_1} - J_{23}^2\omega_1\right)F_1''\Big] - 2\omega_1 J_{21}(J_{13}F_3'+2J_{23}F_3'') - \omega_1 J_{13}(J_{13}F_4+J_{23}F_4'')\omega_1$$
$$+ \left(\frac{J_{21}^2}{\omega_1} - J_{23}^2\omega_1\right)F_1''\Bigg\} \tag{65}$$

$$r_4 = \frac{1}{3\omega_2^2(4\omega_2^2-\omega_1^2)}\Bigg\{8\omega_2^3 J_{22}(J_{14}F_1'+2J_{24}F_1'') - 4\omega_2^2\Big[(J_{14}F_2+J_{24}F_2'')J_{14}\omega_2$$
$$- \left(\frac{J_{22}^2}{\omega_2} - J_{24}^2\omega_2\right)F_2''\Big] - 2\omega_2 J_{22}(J_{14}F_3'+2J_{24}F_3'') - \omega_2 J_{14}(J_{14}F_4+J_{24}F_4'')\omega_2$$
$$- \left(\frac{J_{22}^2}{\omega_2} - J_{24}^2\omega_2\right)F_4''\Bigg\} \tag{66}$$

$$r_5 = \frac{1}{\omega_1\omega_2(2\omega_1+\omega_2)(4\omega_1+2\omega_2)}\Bigg\{(\omega_1+\omega_2)^3\Big[\big\{J_{13}J_{22}(\frac{\omega_1}{\omega_2})^{1/2} - J_{14}J_{21}(\frac{\omega_2}{\omega_1})^{1/2}\big\}F_1'$$
$$-2\big\{J_{21}J_{24}(\frac{\omega_2}{\omega_1})^{1/2} - J_{22}J_{23}(\frac{\omega_1}{\omega_2})^{1/2}\big\}F_1''\Big] - (\omega_1+\omega_2)^2\Big[\big\{2\big\{J_{13}J_{14}F_2$$
$$+(J_{13}J_{24}+J_{14}J_{23})F_2'\big\}(\omega_1\omega_2)^{1/2} + \big\{\frac{J_{21}J_{22}}{(\omega_1\omega_2)^{1/2}} + J_{23}J_{24}(\omega_1\omega_2)^{1/2}\big\}F_2''\Big]$$
$$-(\omega_1+\omega_2)\Big[\big\{J_{13}J_{22}(\frac{\omega_1}{\omega_2})^{1/2} - J_{14}J_{21}(\frac{\omega_2}{\omega_1})^{1/2}\big\}F_3' - 2\big\{J_{21}J_{24}(\frac{\omega_2}{\omega_1})^{1/2}$$
$$- J_{22}J_{23}(\frac{\omega_1}{\omega_2})^{1/2}\big\}F_3''\Big] + \Big[\big\{2\big\{J_{13}J_{14}F_4 + (J_{13}J_{24}+J_{14}J_{23})F_4'\big\}(\omega_1\omega_2)^{1/2}$$
$$+ 2\big\{\frac{J_{21}J_{22}}{(\omega_1\omega_2)^{1/2}} + J_{23}J_{24}(\omega_1\omega_2)^{1/2}\big\}F_4''\Big]\Bigg\} \tag{67}$$

$$r_6 = \frac{-1}{\omega_1\omega_2(2\omega_1-\omega_2)(4\omega_1-2\omega_2)}\Bigg\{(\omega_1-\omega_2)^3\Big[\big\{J_{13}J_{22}(\frac{\omega_1}{\omega_2})^{1/2} - J_{14}J_{21}(\frac{\omega_2}{\omega_1})^{1/2}\big\}F_1'$$
$$+2\big\{J_{21}J_{24}(\frac{\omega_2}{\omega_1})^{1/2} + J_{22}J_{23}(\frac{\omega_1}{\omega_2})^{1/2}\big\}F_1''\Big]$$
$$+(\omega_1-\omega_2)^2\Big[\big\{2\big\{J_{13}J_{14}F_2 + (J_{13}J_{24}+J_{14}J_{23})F_2'\big\}(\omega_1\omega_2)^{1/2}$$
$$-2\big\{\frac{J_{21}J_{22}}{(\omega_1\omega_2)^{1/2}} - J_{23}J_{24}(\omega_1\omega_2)^{1/2}\big\}F_2''\Big] - (\omega_1-\omega_2)\Big[\big\{J_{13}J_{22}(\frac{\omega_1}{\omega_2})^{1/2}$$

$$-J_{14}J_{21}(\frac{\omega_2}{\omega_1})^{1/2}\big\}F_3' + 2\big\{J_{21}J_{22}(\frac{\omega_2}{\omega_1})^{1/2} + J_{22}J_{23}(\frac{\omega_1}{\omega_2})^{1/2}\big\}F_3''\Big]$$
$$-\Big[\big\{2\big\{J_{13}J_{14}F_4 + (J_{13}J_{24} + J_{14}J_{23})F_4'\big\}(\omega_1\omega_2)^{1/2}$$
$$-2\big\{\frac{J_{21}J_{22}}{(\omega_1\omega_2)^{1/2}} - J_{23}J_{24}(\omega_1\omega_2)^{1/2}\big\}F_4''\Big]\Bigg\} \tag{68}$$

$$r_7 = \frac{1}{3\omega_1^2(4\omega_1^2-\omega_2^2)}\Bigg\{8\omega_1^3\Big[J_{13}(J_{13}F_1 + J_{23}F_1')\omega_1 - \Big(\frac{J_{21}^2}{\omega_1} - J_{23}^2\omega_1\Big)F_1''\Big]$$
$$-2\omega_1\Big[\omega_1 J_{13}(J_{13}F_3 + J_{23}F_3') - \Big(\frac{J_{21}^2}{\omega_1} - J_{23}^2\omega_1\Big)F_3''\Big]$$
$$-4\omega_1^2 J_{21}(J_{13}F_2 + J_{23}F_2'')\omega_1 + J_{21}(J_{13}F_4' + 2J_{23}F_4'')\Bigg\} \tag{69}$$

$$r_8 = \frac{-1}{3\omega_2^2(4\omega_2^2-\omega_1^2)}\Bigg\{8\omega_2^3\Big[J_{14}(J_{14}F_1 + J_{24}F_1')\omega_2 - \Big(\frac{J_{22}^2}{\omega_2} - J_{24}^2\omega_2\Big)F_1''\Big]$$
$$+4\omega_2^2 J_{22}(J_{14}F_2 + 2J_{24}F_2'')\omega_2 - 2\omega_2\Big[\omega_2 J_{14}(J_{14}F_3 + J_{24}F_3')$$
$$-\Big(\frac{J_{22}^2}{\omega_2} - J_{24}^2\omega_2\Big)F_3''\Big] - J_{22}(J_{14}F_4' + 2J_{24}F_4'')\Bigg\} \tag{70}$$

$$r_9 = \frac{1}{\omega_1\omega_2(2\omega_1+\omega_2)(\omega_1+2\omega_2)}\Bigg\{(\omega_1+\omega_2)^3\Big[\big\{2J_{13}J_{14}F_1$$
$$+(J_{13}J_{24} + J_{14}J_{23})F_1'\big\}(\omega_1\omega_2)^{1/2} + 2\big\{\frac{J_{21}J_{22}}{(\omega_1\omega_2)^{1/2}} + J_{23}J_{24}(\omega_1\omega_2)^{1/2}\big\}F_1''\Big]$$
$$-(\omega_1+\omega_2)^2\Big[\big\{J_{13}J_{22}(\frac{\omega_1}{\omega_2})^{1/2} - J_{14}J_{21}(\frac{\omega_2}{\omega_1})^{1/2}\big\}F_2'$$
$$-2\big\{J_{21}J_{24}(\frac{\omega_2}{\omega_1})^{1/2} - J_{22}J_{23}(\frac{\omega_1}{\omega_2})^{1/2}\big\}F_2''\Big]$$
$$-(\omega_1+\omega_2)\Big[\big\{2\big\{J_{13}J_{14}F_3 + (J_{13}J_{24} + J_{14}J_{23})F_3'\big\}(\omega_1\omega_2)^{1/2}$$
$$+2\big\{\frac{J_{21}J_{22}}{(\omega_1\omega_2)^{1/2}} + J_{23}J_{24}(\omega_1\omega_2)^{1/2}\big\}F_3''\Big] - \Big[\big\{J_{13}J_{22}(\frac{\omega_1}{\omega_2})^{1/2}$$
$$-J_{14}J_{21}(\frac{\omega_2}{\omega_1})^{1/2}\big\}F_4' - 2\big\{J_{21}J_{24}(\frac{\omega_2}{\omega_1})^{1/2} - J_{22}J_{23}(\frac{\omega_1}{\omega_2})^{1/2}\big\}F_4''\Big]\Bigg\} \tag{71}$$

$$r_{10} = \frac{1}{\omega_1\omega_2(2\omega_1-\omega_2)(2\omega_2-\omega_1)}\Bigg\{(\omega_1-\omega_2)^3\Big[\big\{2J_{13}J_{14}F_1$$
$$+(J_{13}J_{24}+J_{14}J_{23})F_1'\big\}(\omega_1\omega_2)^{1/2} - 2\big\{\frac{J_{21}J_{22}}{(\omega_1\omega_2)^{1/2}} - J_{23}J_{24}(\omega_1\omega_2)^{1/2}\big\}F_1''\Big]$$
$$-(\omega_1-\omega_2)^2\Big[\big\{J_{13}J_{22}(\frac{\omega_1}{\omega_2})^{1/2} - J_{14}J_{21}(\frac{\omega_2}{\omega_1})^{1/2}\big\}F_2'$$
$$+2\big\{J_{21}J_{24}(\frac{\omega_2}{\omega_1})^{1/2} + J_{22}J_{23}(\frac{\omega_1}{\omega_2})^{1/2}\big\}F_2''\Big]$$
$$-(\omega_1-\omega_2)\Big[\big\{2\big\{J_{13}J_{14}F_3 + (J_{13}J_{24}+J_{14}J_{23})F_3'\big\}(\omega_1\omega_2)^{1/2}$$
$$-2\big\{\frac{J_{21}J_{22}}{(\omega_1\omega_2)^{1/2}} - J_{23}J_{24}(\omega_1\omega_2)^{1/2}\big\}F_3''\Big]$$
$$+\Big[\big\{J_{13}J_{22}(\frac{\omega_1}{\omega_2})^{1/2} - J_{14}J_{21}(\frac{\omega_2}{\omega_1})^{1/2}\big\}F_4'$$
$$+2\big\{J_{21}J_{24}(\frac{\omega_2}{\omega_1})^{1/2} - J_{22}J_{23}(\frac{\omega_1}{\omega_2})^{1/2}\big\}F_4''\Big]\Bigg\}. \tag{72}$$

We can write expressions of s_i with the help of r_i replacing F_i by G_i, F_i' by G_i' and F_i'' by G_i'',$(i = 1, 2, 3, 4)$, where

$$F_1 = \frac{-nW_1\epsilon}{6} \tag{73}$$

$$F_2 = \frac{3}{32}\Bigg[\frac{16}{3}\epsilon + 6A_2 - \frac{979}{18}A_2\epsilon + \frac{(143+9\gamma)}{6\sqrt{3}}nW_1 + \frac{(555+376\gamma)}{27\sqrt{3}}nW_1\epsilon$$
$$+\gamma\Bigg\{14 + \frac{4\epsilon}{3} + 25A_2 - \frac{1507}{18}A_2\epsilon - \frac{(215+29\gamma)}{6\sqrt{3}}nW_1$$
$$-\frac{2(1174+169\gamma)}{27\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \tag{74}$$

$$F_3 = \frac{3\sqrt{3}}{16}\Bigg[14 - \frac{16}{3}\epsilon + \frac{23A_2}{2} - \frac{104}{9}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{2(439-68\gamma)}{27\sqrt{3}}nW_1\epsilon$$
$$+\gamma\Bigg\{\frac{32\epsilon}{3} + 40A_2 - \frac{310}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(2519-249\gamma)}{27\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \tag{75}$$

$$F_4 = \frac{-3}{256}\Bigg[364 + 420A_2 - \frac{17801A_2}{9}A_2\epsilon + \frac{(2821+189\gamma)}{3\sqrt{3}}nW_1 - \frac{(23077+9592\gamma)}{27\sqrt{3}}nW_1\epsilon + 28\gamma\Bigg\{23 + \frac{100\epsilon}{21} + \frac{849A_2}{14} + \frac{59}{7}A_2\epsilon - \frac{(125+38\gamma)}{6\sqrt{3}}nW_1 - \frac{(87613-213\gamma)}{27\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \quad (76)$$

$$F_1' = \frac{nW_1\epsilon}{3\sqrt{3}} \quad (77)$$

$$F_2' = \frac{3\sqrt{3}}{16}\Bigg[14 - \frac{16}{3}\epsilon + A_2 - \frac{1367}{18}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{(863-136\gamma)}{27\sqrt{3}}nW_1\epsilon + \gamma\Bigg\{\frac{32\epsilon}{3} + 40A_2 - \frac{382}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(2519-24\gamma)}{27\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \quad (78)$$

$$F_3' = \frac{-9}{8}\Bigg[\frac{8}{3}\epsilon + \frac{203A_2}{6} - \frac{721}{54}A_2\epsilon - \frac{(105+15\gamma)}{18\sqrt{3}}nW_1 - \frac{(319-114\gamma)}{81\sqrt{3}}nW_1\epsilon + \gamma\Bigg\{2 - \frac{4\epsilon}{9} - \frac{173A_2}{6} - \frac{781}{9}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1 - \frac{(265-32\gamma)}{81\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \quad (79)$$

$$F_4' = \frac{-3\sqrt{3}}{16}\Bigg[392 - \frac{532\epsilon}{3} + \frac{1918A_2}{3} - \frac{28582A_2}{9}A_2\epsilon + \frac{(203+1211\gamma)}{9\sqrt{3}}nW_1 + \frac{(949+4378\gamma)}{27\sqrt{3}}nW_1\epsilon + 28\gamma\Bigg\{\frac{108\epsilon}{7} + \frac{4037A_2}{84} - \frac{611}{21}A_2\epsilon + \frac{(8397+919\gamma)}{84\sqrt{3}}nW_1 - \frac{(92266-1869\gamma)}{27\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \quad (80)$$

$$F_1'' = \frac{nW_1\epsilon}{6} \quad (81)$$

$$F_2'' = \frac{-9}{32}\Bigg[\frac{8}{3}\epsilon + \frac{203A_2}{6} - \frac{625}{54}A_2\epsilon - \frac{(105+15\gamma)}{18\sqrt{3}}nW_1 - \frac{(307-114\gamma)}{81\sqrt{3}}nW_1\epsilon + \gamma\Bigg\{2 - \frac{4\epsilon}{9} + \frac{55A_2}{2} - \frac{797}{54}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1 - \frac{(211-32\gamma)}{81\sqrt{3}}nW_1\epsilon\Bigg\}\Bigg] \quad (82)$$

$$F_3'' = \frac{-9\sqrt{3}}{16}\left[2 - \frac{8}{3}\epsilon + \frac{55A_2}{6} - \frac{134}{3}A_2\epsilon - \frac{(37+\gamma)}{18\sqrt{3}}nW_1 - \frac{(93+226\gamma)}{81\sqrt{3}}nW_1\epsilon + \gamma\left\{4\epsilon + \frac{169}{27}A_2\epsilon + \frac{(241+45\gamma)}{18\sqrt{3}}nW_1 - \frac{(1558-126\gamma)}{81\sqrt{3}}nW_1\epsilon\right\}\right] \quad (83)$$

$$F_4'' = \frac{9}{256}\left[\frac{212}{3}\epsilon + \frac{2950A_2}{3} - \frac{1370A_2}{27}A_2\epsilon - \frac{(771+237\gamma)}{9\sqrt{3}}nW_1 - \frac{2(1907-984\gamma)}{81\sqrt{3}}nW_1\epsilon + 28\gamma\left\{\frac{11}{7} + \frac{4\epsilon}{9} - \frac{152A_2}{7} - \frac{36965}{504}A_2\epsilon + \frac{(2569+277\gamma)}{252\sqrt{3}}nW_1 + \frac{(22603+4396\gamma)}{1134\sqrt{3}}nW_1\epsilon\right\}\right] \quad (84)$$

$$G_1 = \frac{-nW_1\epsilon}{6} \quad (85)$$

$$G_2 = \frac{3}{32}\left[14 - \frac{16}{3}\epsilon + A_2 - \frac{1367}{18}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{(863-136\gamma)}{27\sqrt{3}}nW_1\epsilon + \gamma\left\{\frac{32\epsilon}{3} + 40A_2 - \frac{382}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(2519-24\gamma)}{27\sqrt{3}}nW_1\epsilon\right\}\right] \quad (86)$$

$$G_3 = \frac{3\sqrt{3}}{16}\left[\frac{16}{3}\epsilon + 6A_2 - \frac{907A_2}{18}A_2\epsilon + \frac{(143+9\gamma)}{6\sqrt{3}}nW_1 + \frac{(477+403\gamma)}{27\sqrt{3}}nW_1\epsilon + \gamma\left\{14 + \frac{4\epsilon}{3} + \frac{71A_2}{2} - \frac{1489}{18}A_2\epsilon - \frac{(215+29\gamma)}{6\sqrt{3}}nW_1 - \frac{2(1174+169\gamma)}{27\sqrt{3}}nW_1\epsilon\right\}\right] \quad (87)$$

$$G_4 = \frac{3\sqrt{3}}{256}\left[84 + 52\epsilon + 212A_2 - 267A_2\epsilon + \frac{2(299+61\gamma)}{3\sqrt{3}}nW_1 - \frac{(14854+225\gamma)}{27\sqrt{3}}nW_1\epsilon + \gamma\left\{32\epsilon + 156A_2 + 649A_2\epsilon - \frac{(562+8\gamma)}{3\sqrt{3}}nW_1 + \frac{(13285+5169\gamma)}{27\sqrt{3}}nW_1\epsilon\right\}\right] \quad (88)$$

$$G_1' = \frac{-nW_1\epsilon}{\sqrt{3}} \quad (89)$$

$$G_2' = \frac{9}{16}\left[\frac{8}{3}\epsilon + \frac{203A_2}{6} - \frac{625}{54}A_2\epsilon - \frac{(105+15\gamma)}{18\sqrt{3}}nW_1 - \frac{(307-114\gamma)}{81\sqrt{3}}nW_1\epsilon \right.$$
$$-\gamma\left\{2 - \frac{4\epsilon}{9} - \frac{55A_2}{2} - \frac{797}{54}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1\right.$$
$$\left.\left.-\frac{(211-32\gamma)}{81\sqrt{3}}nW_1\epsilon\right\}\right] \quad (90)$$

$$G_3' = \frac{3\sqrt{3}}{8}\left[14 - \frac{16}{3}\epsilon + \frac{65A_2}{6} - \frac{1439}{18}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{(941-118\gamma)}{27\sqrt{3}}nW_1\epsilon\right.$$
$$\left.+\gamma\left\{\frac{32\epsilon}{3} - 40A_2 - \frac{310}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(251-24\gamma)}{27\sqrt{3}}nW_1\epsilon\right\}\right] \quad (91)$$

$$G_4' = \frac{-9}{128}\left[12\epsilon - 287A_2 + \frac{847A_2}{9}A_2\epsilon - \frac{2(28+\gamma)}{\sqrt{3}}nW_1 - \frac{4(2210-69\gamma)}{27\sqrt{3}}nW_1\epsilon\right.$$
$$-\gamma\left\{96 + \frac{152\epsilon}{3} + 135A_2 - \frac{2320}{9}A_2\epsilon + \frac{(497-123\gamma)}{3\sqrt{3}}nW_1\right.$$
$$\left.\left.-\frac{4(17697+32\gamma)}{27\sqrt{3}}nW_1\epsilon\right\}\right] \quad (92)$$

$$G_1'' = \frac{-nW_1\epsilon}{6} \quad (93)$$

$$G_2'' = \frac{9\sqrt{3}}{32}\left[2 - \frac{8}{3}\epsilon + \frac{23A_2}{3} - 44A_2\epsilon - \frac{(37+\gamma)}{18\sqrt{3}}nW_1 - \frac{(123+349\gamma)}{3\sqrt{3}}nW_1\epsilon\right.$$
$$\left.+\gamma\left\{4\epsilon + \frac{88A_2}{27} + \frac{(421+45\gamma)}{18\sqrt{3}}nW_1 - \frac{(1558-126\gamma)}{81\sqrt{3}}nW_1\epsilon\right\}\right] \quad (94)$$

$$G_3'' = \frac{-9}{16}\left[\frac{8}{9}\epsilon + \frac{203A_2}{6} - \frac{589}{54}A_2\epsilon - \frac{5(51+2\gamma)}{18\sqrt{3}}nW_1 - \frac{(349-282\gamma)}{81\sqrt{3}}nW_1\epsilon\right.$$
$$+\gamma\left\{2 - \frac{4\epsilon}{9} - 26A_2 - \frac{412}{27}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1\right.$$
$$\left.\left.-\frac{(211-32\gamma)}{81\sqrt{3}}nW_1\epsilon\right\}\right] \quad (95)$$

$$G_4'' = \frac{-9\sqrt{3}}{256}\Bigg[12 + \frac{20}{3}\epsilon + 76A_2 - \frac{350A_2}{3}A_2\epsilon + \frac{(32\gamma)}{3\sqrt{3}}nW_1 - \frac{2(1529+450\gamma)}{27\sqrt{3}}nW_1\epsilon$$
$$+\gamma\left\{8\epsilon - \frac{749A_2}{3} + \frac{808}{9}A_2\epsilon - \frac{(109-40\gamma)}{3\sqrt{3}}nW_1 + \frac{(35-1269\gamma)}{27\sqrt{3}}nW_1\epsilon\right\}\Bigg] \quad (96)$$

Using transformation $x = B_1^{1,0} + B_2^{1,0}$ and $y = B_1^{0,1} + B_2^{0,1}$ the third order part $H_3 = -L_3$ of the Hamiltonian in $I_1^{1/2}, I_2^{1/2}$ is of the form

$$H_3 = A_{3,0}I_1^{3/2} + A_{2,1}I_1I_2^{1/2} + A_{1,2}I_1^{1/2}I_2 + A_{0,3}I_2^{3/2}. \quad (97)$$

We can verify that in Eq. (97), $A_{3,0}$ vanishes independently as in Deprit & Deprit Bartholomé (1967). Similarly the other coefficients $A_{2,1}$, $A_{1,2}$, $A_{0,3}$ are also found to be zero independently. Hence the third order part H_3 of the Hamiltonian in $I_1^{1/2}$, $I_2^{1/2}$ is zero.

5. Conclusion

Using Whittaker (1965) method we have found that the second order part H_2 of the Hamiltonian is transformed into the normal form $H_2 = \omega_1 I_1 - \omega_2 I_2$. The third order part H_3 of the Hamiltonian in $I_1^{1/2}, I_2^{1/2}$ is zero.

Acknowledgements

We are thankful to D.S.T. Government of India, New Delhi for sanctioning a project DST/MS/140/2K dated 02/01/2004 on this topic. We are also thankful to IUCAA, Pune for providing financial assistance for visiting the library and using the computer facility.

References

Arnold, V. I., 1961, Sov. Math. Dokl., 2, 247
Chernikov, Yu. A., 1970, SvA, 14(1), 176
Deprit, A., & Deprit-Bartholomé, A., 1967, AJ, 72, 173
Ishwar, B., 1997, CeMDA, 65, 253
Kolmogorov, A. N., 1954, DokANSSR, 98, 469
Kushvah, B. S., & Ishwar, B., 2004, Review Bull. Cal. Math. Soc., 12, (1 & 2), 109
Kushvah, B. S., Sharma, J. P., & Ishwar, B., 2006, Review Bull. Cal. Math. Soc., 14, (1 & 2)
Kushvah, B. S., & Ishwar, B., 2006, Journal of Dynamical Systems & Geometric Theories, Vol. 4(1), 79
Liapunov, A. M., 1956. Acad. Sc. USSR

Liou, J. C., Zook, H. A., & Jackson, A. A., 1995, Icar, 116, 186
Maciejewski, A. J., & Gozdziewski, K., 1991, Ap&SS, 179, 1
Mishra, P., & Ishwar, B., 1995, AJ, 110(4), 1901
Moser, J., 1962, Nach. Akad. Wiss., Gottingen, Math. Phys., Kl. II, 1, 1
Murray, C.D., 1994, Icar, 112, 465
Niedzielska, Z., 1994, CeMDA, 58, 203
Poynting, J. H., 1903, MNRAS, 64A, 1
Robertson, H. P., 1937, MNRAS, 97, 423
Schuerman, D. W., 1980, ApJ, 238(1), 337
Subba Rao, P. V., & Sharma, R. K., 1997, CeMDA, 65, 291
Tripathi Deepak Kumar, & Ishwar, B., 2005, News Bull. Cal. Math. Soc., 28,(1 & 3)
Whittaker, E. T., 1965, A treatise on the analytical dynamics of particles and rigid bodies, Cambridge University Press, London, 427